

**Fractal structures and the large scale distribution of galaxies <sup>a</sup>**Luciano Pietronero<sup>1</sup> and Francesco Sylos Labini<sup>2,1</sup><sup>1</sup> *INFN Sezione Roma1, Dip. di Fisica, Università "La Sapienza", P.le A. Moro, 2,  
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Galaxy structures are certainly fractal up to a certain crossover scale  $\lambda_0$ . A clear determination of such a scale is still missing. Usually the conceptual and practical implications of this property are neglected and the structures are only discussed in terms of their global amplitude. Here we present a compact summary of these implications. First, we discuss the problem of the identification of the crossover scale  $\lambda_0$  and the proper characterization of the scaling. We then consider the implications of these properties with respect to various physical phenomena and to the corresponding characteristic values, i.e.  $\gamma_0$ ,  $\sigma_8$ ,  $\Omega$ , etc. These implications crucially depend on the value of  $\lambda_0$ , but they are still important for a relatively small value, say  $\lambda_0 \approx 50h^{-1} Mpc$ . Finally we consider the main theoretical consequences of these results.

**1 Introduction**

Nowadays there is a general agreement about the fact that galactic structures are fractal up to a distance scale of  $\sim 50h^{-1} Mpc$ <sup>46,30</sup> and the increasing interest about the fractal versus homogeneous distribution of galaxy in the last year<sup>10,42,49,6,34,28,7,32,36</sup> has focused, mainly on the determination of the homogeneous scale  $\lambda_0$ .<sup>b</sup> The main point in this discussion is that galaxy structures are fractal no matter what is the crossover scale, and this fact has never been properly appreciated. Clearly, qualitatively different implications are related to different values of  $\lambda_0$ .

- *Characterization of scaling properties.*

Given a distribution of points, the first main question concerns the possibility of defining a physically meaningful average density. In fractal-like systems such a quantity depends on the size of the sample, and it does not represent a reference value, as in the case of an homogeneous distribution. Basically a system cannot be homogeneous below the scale of the maximum void present in a given sample. However the complete

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<sup>b</sup>See the web page <http://pu.phys.uniroma1.it/debate.html> where all these materials have been collected

statistical characterization of highly irregular structures is the objective of Fractal Geometry<sup>33</sup>.

The major problem from the point of view of data analysis is to use statistical methods which are able to properly characterize scale invariant distributions, and hence which are also suitable to characterize an eventual crossover to homogeneity. Our main contribution<sup>41,8,46</sup>, in this respect, has been to clarify that the usual statistical methods (correlation function, power spectrum, etc.) are based on the assumption of homogeneity and hence are not appropriate to test it. Instead, we have introduced and developed various statistical tools which are able to test whether a distribution is homogeneous or fractal, and to correctly characterize the scale-invariant properties. Such a discussion is clearly relevant also for the interpretation of the properties of artificial simulations. The agreement about the methods to be used for the analysis of future surveys such as the Sloan Digital Sky Survey (SDSS) and the two degrees Fields (2dF) is clearly a fundamental issue.

Then, if and only if the average density is found to be not sample-size dependent, one may study the statistical properties of the fluctuations with respect to the average density itself. In this second case one can study basically two different length scales. The first one is the homogeneity scale ( $\lambda_0$ ), which defines the scale beyond which the density fluctuations become to have a small amplitude with respect to the average density ( $\delta\rho < \rho$ ). The second scale is related to the typical length scale of the structures of the density fluctuations, and, according to the terminology used in statistical mechanics<sup>27</sup>, it is called correlation length  $r_c$ . Such a scale has nothing to do with the so-called "correlation length" used in cosmology and corresponding to the scale  $\xi(r_0) = 1$ <sup>39</sup>, which is instead related to  $\lambda_0$  if such a scale exists.

- *Implication of the fractal structure up to scale  $\lambda_0$ .*

The fact that galactic structures are fractal, no matter what is the homogeneity scale  $\lambda_0$ , has deep implication on the interpretation of several phenomena such as the luminosity bias, the mismatch galaxy-cluster, the determination of the average density, the separation of linear and non-linear scales, etc., and on the theoretical concepts used to study such properties. We discuss in detail some of these points. We then review some of the main consequences of the power law behavior of the galaxy number density, by relating various observational quantities (e.g.  $r_0$ ,  $\sigma_8$ ,  $\Omega$ , etc.) to the length scale  $\lambda_0$ .

We also note that the properties of dark matter are inferred from the ones of visible matter, and hence they are closely related. If now one observes different statistical properties for galaxies and clusters, this necessarily implies a change of perspective on the properties of dark matter.

- *Determination of the homogeneity scale  $\lambda_0$ .*

This is, clearly, a very important point which is at the basis of the understanding of galaxy structures and more generally of the cosmological problem. We distinguish here two different approaches: direct tests and indirect tests. By direct tests, we mean the determination of the conditional average density in three dimensional surveys, while with indirect tests we refer to other possible analyses, such as the interpretation of angular surveys, the number counts as a function of magnitude or of distance or, in general, the study of non-average quantities, i.e. when the fractal dimension is estimated without making an average over different observes (or volumes). While in the first case one is able to have a clear and unambiguous answer from the data, in the second one is only able to make some weaker claims about the compatibility of the data with a fractal or a homogeneous distribution. For example the papers of Wu et al.<sup>19</sup> and Nusser & Lahav<sup>36</sup> mainly concern with compatibility arguments, rather than with direct tests. However, also in this second case, it is possible to understand some important properties of the data, and to clarify the role and the limits of some underlying assumptions which are often used without a critical perspective. We do not enter here in the details of the discussion about real data (see e.g. Joyce et al. 1999, Wu et al. 1999), however in the last section we consider separately the case (i)  $\lambda_0 \approx 50h^{-1}Mpc$ , (ii)  $\lambda_0 \approx 300h^{-1}Mpc$  and (iii)  $\lambda_0 \approx 1000h^{-1}Mpc$ , briefly discussing the main theoretical consequences.

## 2 Characterization of scaling properties

In this section we describe in detail the correlation properties of a fractal distribution of points having eventually a crossover towards homogeneity (see Gabrielli & Sylos Labini, 2000 for a more exhaustive discussion of the subject), as the distribution of galaxies is thought to be (see Sylos Labini, Montuori & Pietronero, 1998 and Wu, Lahav & Rees, 1999 for two opposite views on the matter).

Let

$$n(\vec{r}) = \sum_i \delta(\vec{r} - \vec{r}_i) \quad (1)$$

be the number density of points in the system (the index  $i$  runs over all the points) and let us suppose to have an infinite system. If the presence of an object at the point  $\vec{r}_1$  influences the probability of finding another object at  $\vec{r}_2$ , these two points are correlated. Hence there is a correlation at the scale distance  $r$  if

$$G(r) = \langle n(\vec{0})n(\vec{r}) \rangle \neq \langle n \rangle^2 \quad (2)$$

where we average over all occupied points of the system chosen as origin and on the total solid angle, supposing statistical isotropy. On the other hand, there is no correlation if

$$G(r) = \langle n \rangle^2. \quad (3)$$

### 2.1 Homogeneity scale and correlation length

The proper definition of  $\lambda_0$ , the *homogeneity scale*, is the length scale beyond which the average density becomes to be well-defined, i.e. there is a crossover towards homogeneity with a flattening of  $G(r)$ . The length-scale  $\lambda_0$  is related to the typical dimension of the largest voids in the system. On the other hand, the *correlation length*  $r_c$  separates correlated regimes of the fluctuations with respect to the average density from uncorrelated ones, and it can be defined only if a crossover towards homogeneity is shown by the system, i.e. if  $\lambda_0$  exists<sup>27</sup>. In other words  $r_c$  defines the organization in geometrical structures of the fluctuations with respect to the average density. Clearly  $r_c > \lambda_0$ : only if the average density can be defined one may study the correlation length of the fluctuations from it. In the case  $\lambda_0$  is finite and then  $\langle n \rangle > 0$ , in order to study the correlation properties of the fluctuations around the average and then the behaviour of  $r_c$ , we can introduce the correlation function

$$\xi(r) = \frac{\langle n(0)n(r) \rangle - \langle n \rangle^2}{\langle n \rangle^2}. \quad (4)$$

In the case of a fractal distribution, the average density  $\langle n \rangle$  in the **infinite system** is zero, then  $G(r) = 0$  and  $\lambda_0 = \infty$  and consequently  $\xi(r)$  is not defined. In this case the only well defined quantity characterizing the two point correlations is the function  $\gamma(r)$ <sup>41,8</sup>:

$$\gamma(r) = \lim_{R_s \rightarrow \infty} \frac{\langle n(r)n(0) \rangle_{R_s}}{\langle n \rangle_{R_s}} \quad (5)$$

where  $R_s$  is the size of the a generic finite sample of the system,  $\langle \dots \rangle_{R_s}$  indicates the average over all the points of the sample as origins, hence  $\langle n \rangle_{R_s}$  is the average density of the sample. This function measures the average density

of points at a distance  $r$  from another occupied point, and this is the reason why it is called the conditional average density<sup>41</sup>. Obviously in the case of a distribution for which  $\lambda_0$  is finite,  $\rho(r)$  provides the same information of  $G(r)$ , i.e. it characterizes the correlation properties for  $r < \lambda_0$  and the crossover to homogeneity.

A very important point is represented by the kind of information about the correlation properties of the **infinite system** which can be extracted from the analysis of a **finite sample** of it. In Pietronero (1987) it is demonstrated that even in the super-correlated case of a fractal the estimate of  $\rho(r)$  extracted from a finite sample, is not dependent on the sample size  $R_s$ , providing a good approximation of that of the whole system. Clearly this is true a part from statistical fluctuations due to the fact that in a finite sample the average over the all possible origins is an average over a finite number of points, while in the global infinite system the average is over an infinite number of points. In fact,  $\rho(r)$  extracted from a sample can be written in the following way:

$$\rho(r) = \frac{1}{N} \sum_{i=1}^N \frac{1}{4\pi r^2 \Delta r} \int_r^{r+\Delta r} n(\vec{r}_i + \vec{r}') d^3 r', \quad (6)$$

where  $N$  is the number of points in the sample,  $n(\vec{r}_i + \vec{r}')$  is the number of points in the volume element  $d^3 r'$  around the point  $\vec{r}_i + \vec{r}'$  and  $\Delta r$  is the thickness of the shell at distance  $r$  from the point at  $\vec{r}_i$ .

Therefore, from an operative point of view, having a finite sample of points (e.g. galaxy catalogs), the first analysis to be done concerns the determination of  $\rho(r)$  of the sample itself. Such a measurement is necessary to distinguish between the two cases: (1) a crossover towards homogeneity in the sample shown by a flattening of  $\rho(r)$ , and hence an estimate of  $\lambda_0 < R_s$  and  $\langle n \rangle$ ; (2) a continuation of the fractal behavior. Obviously only in the case (1), it is physically meaningful to study the correlation function  $\xi(r)$  (Eq.4), and extract from it the length scale  $r_0$  ( $\xi(r_0) = 1$ ), which is related to the intrinsic homogeneity scale  $\lambda_0$ . The functional behavior of  $\xi(r)$  with distance gives instead information on the correlation length of the density fluctuations.

## 2.2 The case of a fractal distribution ( $R_s \ll \lambda_0$ )

Hereafter we study the three-dimensional case, i.e.  $d = 3$ , and we suppose that the sample is a sphere of radius  $R_s$ . Obviously, this choice is not a restriction.

Let us analyze the case  $R_s \ll \lambda_0 < r_c$ . This is the so called “fractal” case, and it is compatible with both the situation of  $\lambda_0$  finite, but  $R_s \ll \lambda_0$  (a sample-size which is smaller than the homogeneity scale), or the situation in which  $\lambda_0 \rightarrow \infty$ , i.e. the case of a fractal distribution at any scale.

It is simple to show<sup>41,8,46</sup> that in this case (and in a spherical sample), Eq.6 becomes

$$\xi(r) = \frac{BD}{4\pi} r^{D-3} \quad (7)$$

with  $B = N/R_s^D$ . Note that  $B$  is independent on the sample size: in fact, by changing  $R_s$ ,  $N$  in average scales as  $R_s^D$ . This shows the aforementioned assertion that  $\xi(r)$  is practically independent on the sample-size. On the other hand, it is possible to show that  $B$  is related approximatively to the average distance between nearest neighbors points in the system<sup>47</sup>:

$$\ell \approx \left(\frac{1}{B}\right)^{\frac{1}{D}}, \quad e \left(1 + \frac{1}{D}\right) \quad (8)$$

where  $\gamma_e$  is the Euler's gamma-function.

### 2.3 The “standard” correlation function for a fractal distribution

As already mentioned, in the fractal case ( $R_s \ll \lambda_0$ ), the sample estimate of the homogeneity scale, through the value of  $r$  for which the sample-dependent correlation function  $\xi(r)$  (given by Eq.9) is equal to 1, is meaningless: This estimate is the so-called “correlation length”  $r_0$ <sup>39</sup> in the standard approach of cosmology. As we discuss below,  $r_0$  has nothing to share with the *true* correlation length  $r_c$ . Let us see why  $r_0$  is unphysical in the case  $R_s \ll \lambda_0$ .  $\xi(r)$  is given operatively by

$$\xi(r) = \frac{\langle n(r)n(0) \rangle_{R_s}}{\langle n \rangle_{R_s}^2} - 1 = \frac{\xi(r)}{\langle n \rangle_{R_s}} - 1. \quad (9)$$

The basic point in the present discussion<sup>41</sup>, is that the mean density of the sample,  $\langle n \rangle_{R_s}$ , used in the normalization of  $\xi(r)$ , is not an intrinsic quantity of the system, but it is a function of the finite size  $R_s$  of the sample.

In fact, from Eq.7, the expression of the  $\xi(r)$  of the sample in the case of fractal distributions is<sup>41</sup>

$$\xi(r) = \frac{D}{3} \left(\frac{r}{R_s}\right)^{D-3} - 1. \quad (10)$$

From Eq.10 it follows that  $r_0$  is a linear function of the sample size  $R_s$

$$r_0 = \left(\frac{D}{6}\right)^{\frac{1}{3-D}} R_s \quad (11)$$

and hence it is a spurious quantity without physical meaning but it is simply related to the sample's finite size.

We note that the amplitude of  $\xi(r)$  (Eq.7) is related to the lower cut-off of the fractal  $\ell$  by Eq.8, while the amplitude of  $\xi(r)$  is related to the upper cut-off (sample size  $R_s$ ) of the distribution. This crucial difference has never been appreciated appropriately.

Finally we stress that in the standard analysis of galaxy catalogs the fractal dimension is estimated by fitting  $\xi(r)$  with a power law, which instead, as one can see from Eq.10, it is power law only for  $r \ll r_0$  (or  $\xi \gg 1$ ). For larger distances there is a clear deviation from the power law behavior due to the definition of  $\xi(r)$ . Again this deviation is due to the finite size of the observational sample and does not correspond to any real change in the correlation properties. It is easy to see that, if one estimates the exponent at distances  $r \lesssim r_0$ , one systematically obtains a higher value of the correlation exponent due to the break of  $\xi(r)$  in a log-log plot. To illustrate more clearly this we compute the log derivative of Eq.10 with respect to  $\log(r)$ , indicating  $D - 3$  with  $\gamma$  and its estimate with  $\gamma'$ :

$$\gamma' = \frac{d(\log(\xi(r)))}{d \log(r)} = \frac{2r_0^\gamma r^{-\gamma}}{2r_0^\gamma r^{-\gamma} - 1} \gamma, \quad (12)$$

where  $r_0$  is defined by Eq.11. The tangent to  $\xi(r)$  at  $r = r_0$  has a slope  $\gamma' = 2\gamma$ . This explain why it has been found in galaxy and cluster catalogs that  $\gamma \sim 2$  by the  $\xi(r)$  analysis<sup>12,14,49</sup> instead of  $\gamma \sim 1$  found with the  $\xi(r)$  analysis<sup>46</sup>.

#### 2.4 The case of a fractal distribution with a crossover to homogeneity ( $R_s \gg \lambda_0$ )

Let us now analyze the case of a fractal with a crossover to homogeneity. In Coleman & Pietronero (1992) a very simple approximation has been used to describe such a situation which we discuss in more detail below.

By defining  $\langle n \rangle_{R_s} = N/V$  with  $V = 4\pi R_s^3/3$ , it is simple to see<sup>8</sup> that the behavior of  $\xi(r)$  in our sample is fractal (i.e.  $\xi(r)$  is a power law) up to a certain distance  $\lambda_0$ , and then it flattens, becoming homogeneous at scales  $R_s > r \gg \lambda_0$ :

$$\left\{ \begin{array}{l} \xi(r) = \frac{DB}{4\pi} r^{D-3} \text{ for } \ell \leq r \ll \lambda_0 \\ \xi(r) \simeq \langle n \rangle_{R_s} \text{ for } \lambda_0 \ll r \leq R_s, \end{array} \right. \quad (13)$$

where  $\langle n \rangle_{R_s}$  is the estimation of the average density in the sample of size  $R_s$ . That is,  $\langle n \rangle_{R_s}$  does not depend on  $r$  if  $\lambda_0 \ll r \lesssim R_s$ , apart small amplitude fluctuations. In Eq.13 the detailed approach to homogeneity depends on the

specific properties of the fluctuations around the average density, i.e. it is determined by  $r_c$ . Hence, the statistical properties of the density fluctuations determine how good is the estimation of the average density through  $\langle n \rangle_{R_s}$ .

From the definition of the function  $\xi(r)$  we can find<sup>6</sup>

$$\xi(r) \approx \left( \frac{r}{\lambda_0} \right)^{D-3} f \left( \frac{r}{r_c} \right). \quad (14)$$

Note that the amplitude of  $\xi(r)$  is determined by the homogeneity scale  $\lambda_0$  which has been previously extracted from  $\rho(r)$ , and that in this approximation  $r_0 \sim \lambda_0$ . The function  $\xi(r)$  characterizes the correlations among the fluctuations of the distribution with respect to the average density. It is important to clarify that these fluctuations must be both positive and negative, in fact the integral over the whole sample of Eq.9 must be 0. Hence the function  $f \left( \frac{r}{r_c} \right)$  must be oscillating, and in the case  $\lambda_0 < r_c, R_s$ , it should present an exponential cut-off at  $r \approx r_c$ . We have that, when  $\xi(\lambda_0) \simeq 1$ , the density fluctuations begin to become small with respect to the average density  $\langle n \rangle$ , but if  $\lambda_0 < r < r_c$  they are still well correlated among them. Only for  $r \gg r_c$  the fluctuations are not correlated.

Let us now consider the case  $\lambda_0 \ll R_s < r_c$ . This situation is compatible with the following two situations:  $r_c$  finite, but larger than  $R_s$ , and the case  $r_c \rightarrow \infty$ . In both cases,  $\xi(r)$  of our sample should be a power law modulated by an oscillating function  $g(r)$  which describes the positive and negative fluctuations with respect to the average density,

$$\xi(r) = \left( \frac{r}{\lambda_0} \right)^{-\gamma} g(r). \quad (15)$$

In such a situation the (positive and negative) fluctuations from the average density are of all sizes and they do not have any intrinsic characteristic scale: this is a critical system (see Gaite et al., 1999 for a more detailed discussion). The only intrinsic scale of the system is then  $\lambda_0$ , the length-scale beyond which  $\rho(r)$  flattens and the fluctuations are small with respect to the average.

Let us suppose to be in the case in which  $\rho(r)$  flattens at a certain  $\lambda_0 \ll R_s$ . We can then evaluate the correlation function  $\xi(r)$  of the sample via Eq.9. At this point we can clarify how to interpret the eventual cut-off shown by  $\xi(r)$ .

- If the cut-off scale is well below  $R_s$ , we can be sure that it is a good estimate of the intrinsic correlation length  $r_c$ ;
- if the cut-off is at a scale  $r \simeq R_s$  we can have two cases: it represents an “intrinsic” cut-off with  $r_c \simeq R_s$ , or it is only a finite-size effect due



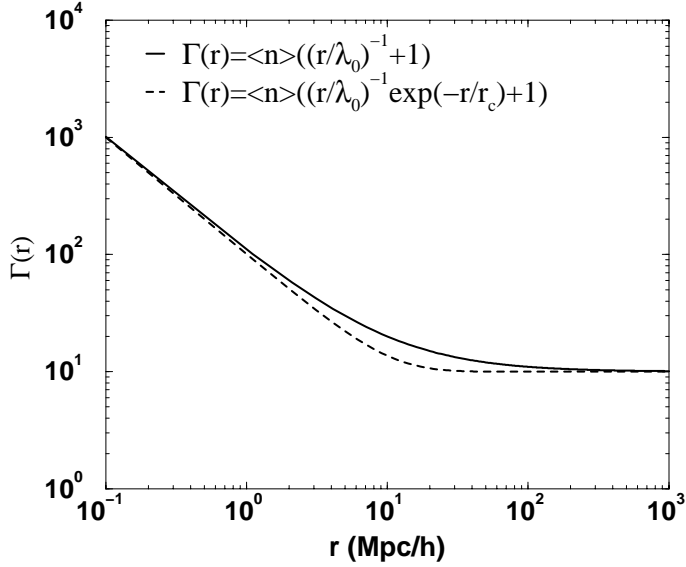


Figure 1: The conditional average density  $\Gamma(r)$  for a distribution which has a power law behavior at small scales ( $r < \lambda_0$ ), followed by a transition to homogeneity at the scale  $\lambda_0 = 10h^{-1}Mpc$ . The behavior of the flattening depends on the correlation properties of the density fluctuations, i.e. on the functional behavior of  $\xi(r)$ . The dotted line corresponds to a system which has a finite correlation length  $r_c = 30h^{-1}Mpc$ , while the solid line describes a system whose density fluctuations present correlation over all scales. From the  $\Gamma(r)$ -analysis alone it is possible to compute  $\lambda_0$  but not  $r_c$ .

to the fact that from Eq.9  $|\xi(R_s)| = 0$ . In order to distinguish between these two possibilities, it is necessary to increase the sample size and to look at the behavior of the cut-off scale. If it increases proportionally to  $R_s$ , then it is a finite size effect. Otherwise if it does not change, it represents the estimate of the intrinsic correlation length  $r_c$ .

In Fig.1 we show two possible behaviours of the flattening of  $\Gamma(r)$ , while in Fig.2 it is shown the corresponding  $\xi(r)$  (we neglect for simplicity the oscillating term which must be present, and we have considered the situation  $R_s \rightarrow \infty$ ).

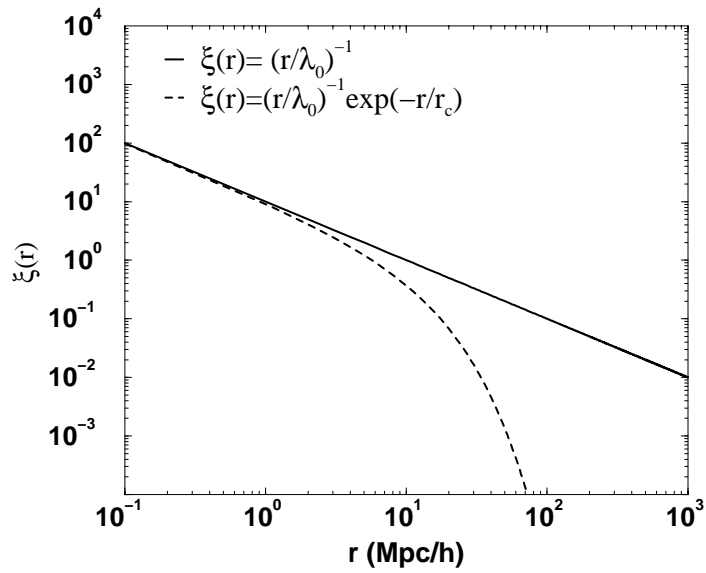


Figure 2: The  $\xi(r)$  correlation function for the distributions shown in the previous figure. With this analysis it is possible to compute the correlation length  $r_c$  (finite or infinite) of the density fluctuations. The dotted line corresponds to an exponential decay, and hence to a finite value of the correlation length ( $r_c = 30h^{-1}Mpc$ ) while the solid line corresponds to an infinite correlation length, and hence to a power law behavior. The length scale at which  $\xi(r_0) = 1$  gives a reliable estimation of  $\lambda_0$ .

## 2.5 About the amplitude of $\xi(r)$

We note that if  $\lambda_0 \ll R_s$ ,  $\lambda_0$  has nothing to share with questions like “which is the typical size of structures in the system?” or “up to which length-scale the system is clusterised?”. The answer to this question is strictly related to  $r_c$  and not to  $\lambda_0$ . The length scale  $r_c$  characterizes the distance over which two different points are correlated (clusterised)<sup>27</sup>. In fact, this property is not related to how large are the fluctuations with respect to the average ( $\lambda_0$ ), but to the length extension of their correlations ( $r_c$ ).

To be more specific, let us consider a fixed set of density fluctuations. They can be superimposed to different values of a uniform density background. The larger is this background the lower  $\lambda_0$ , but obviously the length scale of the correlations ( $r_c$ ) among these fluctuations is not changed, i.e. they are clusterised independently of the background.

One can see<sup>26,25</sup> that a linear amplification of  $\xi(r)$  such that

$$\xi'(r) = A\xi(r) \quad (16)$$

doesn't change  $r_c$  (which can be finite or infinite) but only  $\lambda_0$ , i.e. if  $A > 1$  we need larger subsamples to have a good estimation of  $\langle n \rangle$ , but the characteristic length (correlation length) of the structures is not changed.

## 2.6 Homogeneity scale and the size of voids

Basically  $\lambda_0$  is related to the maximum size of voids: the average density will be constant, at least, on scales larger than the maximum void in a given sample. Several authors have approached this problem by looking at voids distribution. For example El-Ad and Piran (1997) have shown that the SSRS2 and IRAS 1.2 Jy. redshift surveys are dominated by voids: they cover the  $\sim 50\%$  of the volume. Moreover the two samples show very similar properties even if the IRAS voids are  $\sim 33\%$  larger than SSRS2 ones because they are not bounded by narrow angular limits as the SSRS2 voids. The voids have a scale of at least  $\sim 40 \div 50 h^{-1} Mpc$  and the largest void in the SSRS2 sample has a diameter of  $\sim 60 h^{-1} Mpc$ , i.e. comparable to the Bootes void. The problem is to understand whether such a scale has been fixed by the samples' volume, or whether there is a tendency not to find larger voids: in this case one would have a (weaker evidence) for the homogeneity scale. In any case, we note that the homogeneity scale cannot be smaller than the scale of the largest void found in these samples and that one has to be very careful when comparing the size of the voids to the effective depth of catalogs. For example in the Las Campanas Redshift Survey, even if it is possible to extract sub-samples limited at  $\sim 500 h^{-1} Mpc$ , the volume of space investigated is not so large, as

the survey is made by thin slices. In such a situation a definitive answer to the dimension of the of voids, and hence to the existence of the homogeneity scale, is rather difficult and uncertain.

## 2.7 Luminosity Bias

We would like to stress again that, even if the fractal behavior breaks at a certain scale  $\lambda_0$ , the use of  $\xi(r)$  is in anyhow inconsistent at scales smaller than  $\lambda_0$ . We illustrate below an example of the confusion due to the use of  $\xi(r)$  when  $r \ll \lambda_0$ .

From the use of the  $\xi(r)$  analysis, it has been found that  $r_0$  is different in different volume (hereafter VL) samples. In particular it has been found<sup>4</sup> that deeper is the VL sample, larger is the value of  $r_0$ . As the deeper VL samples contain brighter galaxies, this fact has been interpreted as a real physical phenomenon, leading to the idea that more brighter galaxies are more strongly clustered than fainter ones, in view of their larger correlation amplitude: this is the so-called luminosity segregation phenomenon<sup>13,37</sup>. In other words, the fact that the giant galaxies are "more clustered" than the dwarf ones, i.e. that they are located in the peaks of the density field, has given rise to the proposition that larger objects may correlate up to larger length scales and that the amplitude of  $\xi(r)$  is larger for giants than for dwarfs one. The deeper VL samples contain galaxies which are in average brighter than those in the VL samples with smaller depths. As the brighter galaxies should have a larger correlation length the shift of  $r_0$  in different samples can be related, at least partially, with the phenomenon of luminosity segregation.

As previously discussed there are two problems with such a model: (i) The amplitude of  $\xi(r)$  in an homogeneous distribution, does not give any information about the clustering "strength". It is instead related to the local amplitude of the fluctuations with respect to the average density. (ii) The amplitude of  $\xi(r)$  has a physical meaning only in the case  $\lambda_0$  is found to be finite and smaller than the sample's size. This is clearly not the case up, at least, to  $\sim 50h^{-1}Mpc$ .

A natural explanation of the scaling of  $r_0$  is then the fractal behavior of galaxy distribution, and more specifically the fact that  $r_0$  is a fraction of the sample's size in the fractal case. The fact that giant elliptical galaxies are located in the core of rich clusters, and other morphological properties of this kind, can be naturally related to the multifractal properties of matter distribution<sup>8,45</sup>. In such a case, bright galaxies are more strongly clustered than fainter ones in view of the fact that their fractal dimension is smaller<sup>26</sup>.

## 2.8 Power Spectrum of density fluctuations

The problems with the standard correlation analysis also show that the properties of fractal correlations have not been really appreciated. These problems are actually far more serious and fundamental than mentioned, for example, by Landy<sup>32</sup> and the idea that they can be solved by simply taking the Fourier transform is once more a proof of the superficiality of the discussion. We have extensively shown<sup>44,46</sup> that the power spectrum of the density fluctuations has the same kind of problems which  $\xi(r)$  has, because it is normalized to the average density as well. The density contrast  $\delta(r) = \delta\rho(r)/\langle\rho\rangle$  is not a physical quantity unless the average density is demonstrated to exist. More specifically, like in the case of  $\xi(r)$ , the power spectrum (Fourier Transform of the correlation function) is affected by finite size effects at large scale: even for a fractal distribution the power spectrum has not a power law behavior but it shows a large scale (small  $k$ ) cut-off which is due to the finiteness of the sample<sup>44</sup>. Hence the eventual detection of the turnover of the power spectrum, which is expected in CDM-like models to match the galaxy clustering to the anisotropies of the CMBR, must be considered a finite size effect, unless a clear determination of the average density in the *same sample* has been done.

Essentially all the currently elaborated models of galaxy formation<sup>40</sup> *assume large scale homogeneity* and predict that the galaxy power spectrum (hereafter PS), which is *the PS of the density contrast*, decreases both toward small scales and toward large scales, with a turnaround somewhere in the middle, at a scale  $\lambda_f$  that can be taken as separating “small” from “large” scales. Because of the homogeneity assumption, the PS amplitude should be independent on the survey scale, any residual variation being attributed to luminosity bias (or to the fact that the survey scale has not yet reached the homogeneity scale). However, the crucial clue to this picture, the firm determination of the scale  $\lambda_f$ , is still missing, although some surveys do indeed produce a turnaround scale around  $100 h^{-1} Mpc$ <sup>3,22</sup>. Recently, the CfA2 survey analyzed by<sup>37</sup> (hereafter PVGH) (and confirmed by SSRS2<sup>11</sup> - hereafter DVGHP), showed a  $n = -2$  slope up to  $\sim 30 h^{-1} Mpc$ , a milder  $n \approx -1$  slope up to  $200 h^{-1} Mpc$ , and some tentative indication of flattening on even larger scales. PVGH also find that deeper subsamples have higher power amplitude, i.e. that the amplitude scales with the sample depth.

*In the following we argue that both features, bending and scaling, are a manifestation of the finiteness of the survey volume, and that they cannot be interpreted as the convergence to homogeneity, nor to a PS flattening.* The systematic effect of the survey finite size is in fact to suppress power at large scale, mimicking a real flattening. Clearly, this effect occurs whenever galaxies

have not a correlation scale much larger than the survey size, and it has often been studied in the context of standard scenarios<sup>29,9</sup>. We push this argument further, by showing that even a fractal distribution of matter, which never reaches homogeneity, shows a sharp flattening and then a turnaround. Such features are partially corrected, but not quite eliminated, when the correction proposed by<sup>38</sup> is applied to the data. We show also how the amplitude of the PS depends on the survey size as long as the system shows long-range correlations.

The standard PS (SPS) measures directly the contributions of different scales to the galaxy density contrast  $\delta\rho/\rho$ . It is clear that the density contrast, and all the quantities based on it, is meaningful only when one can define a constant density, i.e. reliably identify the sample density with the average density of all the Universe. In other words in *the SPS analysis one assumes that the survey volume is large enough to contain a homogeneous sample*. When this is not true, and we argue that is indeed an incorrect assumption in all the cases investigated so far, a false interpretation of the results may occur, since both the shape and the amplitude of the PS (or correlation function) depend on the survey size.

Let us recall the basic notation of the PS analysis. Following Peebles<sup>39</sup> we imagine that the Universe is periodic in a volume  $V_u$ , with  $V_u$  much larger than the (presumed) maximum homogeneity scale. The survey volume  $V \in V_u$  contains  $N$  galaxies at positions  $\vec{r}_i$ , and the galaxy density contrast is

$$\delta(\vec{r}) = \frac{n(\vec{r})}{\hat{n}} - 1 \quad (17)$$

where it is assumed that exists a well defined constant density  $\hat{n}$ , obtained averaging over a sufficiently large scale. The density function  $n(\vec{r})$  can be described by a sum of delta functions:  $n(\vec{r}) = \sum_{i=1}^N \delta^{(3)}(\vec{r} - \vec{r}_i)$ . Expanding the density contrast in its Fourier components we have

$$\delta_{\vec{k}} = \frac{1}{N} \sum_{j \in V} e^{i\vec{k}\vec{r}_j} - W(\vec{k}), \quad (18)$$

where

$$W(\vec{k}) = \frac{1}{V} \int_V d\vec{r} W(\vec{r}) e^{i\vec{k}\vec{r}} \quad (19)$$

is the Fourier transform of the survey window  $W(\vec{r})$ , defined to be unity inside the survey region, and zero outside. If  $\xi(\vec{r})$  is the correlation function of the galaxies ( $\xi(\vec{r}) = \langle n(\vec{r})n(0) \rangle / \hat{n}^2 - 1$ ), the true PS  $P(\vec{k})$  is defined as the

Fourier conjugate of the correlation function  $\xi(r)$ . Because of isotropy the PS can be simplified to

$$P(k) = 4\pi \int \xi(r) \frac{\sin(kr)}{kr} r^2 dr. \quad (20)$$

The variance of  $\delta_{\vec{k}}$  is<sup>39,38,23</sup>

$$\langle |\delta_{\vec{k}}|^2 \rangle = \frac{1}{N} + \frac{1}{V} \tilde{P}(\vec{k}). \quad (21)$$

The first term is the usual additional shot noise term while the second is the true PS convoluted with a window function which describe the geometry of the sample (PVGH)

$$\tilde{P}(\vec{k}) = \frac{V}{(2\pi)^3} \int \langle |\delta_{\vec{k}'}|^2 \rangle |W(\vec{k} - \vec{k}')|^2 d^3 \vec{k}'. \quad (22)$$

$$\tilde{P}(\vec{k}) = \int d\vec{k}' P(\vec{k}') F(\vec{k} - \vec{k}'), \quad (23)$$

with

$$F(\vec{k} - \vec{k}') = \frac{V}{(2\pi)^3} |W(\vec{k} - \vec{k}')|^2. \quad (24)$$

We apply now this standard analysis to a fractal distribution. We recall the expression of  $\xi(r)$  in this case is

$$\xi(r) = [(3 - \gamma)/3](r/R_s)^{-\gamma} - 1, \quad (25)$$

where  $\gamma = 3 - D$ . A key point of our discussion is that that on scales larger than  $R_s$  the  $\xi(r)$  cannot be calculated without making assumptions on the distribution outside the sampling volume.

As we have already mentioned, in a fractal quantities like  $\xi(r)$  are scale dependent: in particular both the amplitude and the shape of  $\xi(r)$  depend the survey size. It is clear that the same kind of finite size effects are also present when computing the SPS, so that it is very dangerous to identify real physical features induced from the SPS analysis without first a firm determination of the homogeneity scale.

The SPS for a fractal distribution model described by Eq.25 inside a sphere of radius  $R_s$  is

$$P(k) = \int_0^{R_s} 4\pi \frac{\sin(kr)}{kr} \left[ \frac{3 - \gamma}{3} \left( \frac{r}{R_s} \right)^{-\gamma} - 1 \right] r^2 dr = \frac{a_k(R_s) R_s^{3-D}}{k^D} - \frac{b_k(R_s)}{k^3}. \quad (26)$$

Notice that the integral has to be evaluated inside  $R_s$  because we want to compare  $P(k)$  with its *estimate* in a finite size spherical survey of scale  $R_s$ . In the general case, we must deconvolve the window contribution from  $P(k)$ ;  $R_s$  is then a characteristic window scale. Eq.26 shows the two scale-dependent features of the PS. First, the amplitude of the PS depends on the sample depth. Secondly, the shape of the PS is characterized by two scaling regimes: the first one, at high wavenumbers, is related to the fractal dimension of the distribution in real space, while the second one arises only because of the finiteness of the sample. In the case of  $D = 2$  in Eq.26 one has:

$$a_k(R_s) = \frac{4\pi}{3}(2 + \cos(kR_s)) \quad (27)$$

and

$$b_k(R_s) = 4\pi \sin(kR_s) . \quad (28)$$

The PS is then a power-law with exponent  $-2$  at high wavenumbers, it flattens at low wavenumbers and reaches a maximum at  $k \approx 4.3/R_s$ , i.e. at a scale  $\lambda \approx 1.45R_s$ . The scale at which the transition occurs is thus related to the sample depth. In a real survey, things are complicated by the window function, so that the flattening (and the turnaround) scale can only be determined numerically.

In practice one has several complications. First, the survey in general is not spherical. This introduces a coupling with the survey window which is not easy to model analytically. For instance, we found that windows of small angular opening shift to smaller scales the PS turnaround. This is analogous to what happens with the correlation function of a fractal: when it is calculated in small angle surveys, the correlation length  $r_0$  decreases. Second, the observations are in redshift space, rather than in real space. The peculiar velocities generally make steeper the PS slope<sup>23</sup> with respect to the real space. Third, in a fractal the intrinsically high level of fluctuations makes hard a precise comparison with the theory when the fractal under study is composed of a relatively small number of points.

### 3 Implications for cosmology

We now consider some implications for cosmology of the scaling properties of galaxy distribution, up to a length scale  $\lambda_0$ . For example, we consider more specifically  $\lambda_0 \approx 50h^{-1}Mpc$ .

#### 3.1 Estimation of the average luminosity and mass density

From the studies of Large Scale Structures (LSS) of galaxies and galaxy clusters, one would like to estimate the average density of visible matter and then



to infer the one of the whole (visible plus dark, i.e. all the matter in clusterised objects) matter distribution. While for the first we have direct estimations, for the second we have only indirect methods, especially at large scales, based on some assumptions, which can be tested by looking at the distribution of what is observable, i.e. visible matter.

We briefly describe how to do such a measurement in galaxy redshift catalogs directly, from the knowledge of galaxy positions and luminosity (for a more detailed discussion see Sylos Labini, 2000). In this case, and by measuring the Mass-to-Luminosity ratio, one can infer, from the average luminosity density, the average mass density. The new point we address more specifically is that galaxies are fractally distributed up to a certain crossover scale  $\lambda_0$ . As there is still some controversy about the value of  $\lambda_0$ <sup>46,49,7,30,34</sup> we give the estimation as a function of  $\lambda_0$ . We stress that the way this estimation is performed, is substantially different from the usual one<sup>40</sup>, because in such a case the fractal behavior is not considered at all, and one assumes a perfect homogeneous distribution at relatively small scale ( $\lambda_0 \sim 5 \div 10 h^{-1} Mpc$ ). This situation is clearly not the one corresponding to the more "optimistic" estimation of the homogeneity scale  $\lambda_0$ <sup>46,49,30</sup>. The other assumption usually made is that galaxy positions are independent on their luminosity. We have shown<sup>48</sup> that, although such an assumption cannot describe local morphological properties of galaxy distribution<sup>45</sup>, it works rather well in the available galaxy redshift surveys.

The estimation of the the average density we are able to make depends hence on two parameters. The first one is the homogeneity scale  $\lambda_0$  and the second is the Mass-to-Luminosity ratio. We can give an upper limit to  $\Omega$  by taking the highest  $(\mathcal{M}/\mathcal{L})_c = 300h$  observed up to now<sup>1</sup> (in clusters of galaxies) to be universal across all the scales, and by considering a lower limit for the homogeneity scale  $\lambda_0 = 50 h^{-1} Mpc$ . We compute the critical  $(\mathcal{M}/\mathcal{L})_{crit}$ , i.e. the Mass-to-Luminosity ratio needed to have  $\Omega = 1$ . As the others, also this parameter depends on the homogeneity scale  $\lambda_0$ .

### 3.2 Average luminosity density from galaxy catalogs

Let

$$\langle \nu(r, L) \rangle dL d^3r = \phi(L) \langle \nu(r) \rangle dL d^3r = A r^{D-3} L^\alpha e^{-\frac{r}{L_*}} d^3r dL \quad (29)$$

be the average number of galaxies in the volume element  $d^3r$  at distance  $r$  from a observer located on a galaxy, and with luminosity in the range  $[L, L + dL]$ . In Eq.29 we have used the fact that the galaxy luminosity function has been observed to have the so-called Schechter shape with parameters  $L_*$  (luminosity cut-off) and  $\alpha$  (power law index) which can be determined experimentally.

The conditional average space density  $\langle \nu(r) \rangle$  has a power law behavior corresponding to a fractal dimension  $D$  (which eventually can be a function of scale, and hence can approach to  $D = 3$  at a scale  $\lambda_0$ ). Both the fractal dimension  $D$  and the overall amplitude  $A$  can be determined in redshift surveys. Hence  $\langle \nu(r, L) \rangle$  is a function of four parameters:  $L_*, \alpha, D, A$ . Moreover we note that by writing  $\langle \nu(r, L) \rangle$  as a product of the space density and of the luminosity function we have implicitly assumed that galaxy positions are independent on galaxy luminosity.

We would like to estimate the average luminosity density in a sphere of radius  $R$  and volume  $V(R)$  placed around a galaxy, and defined as

$$\langle j(< R) \rangle = \frac{1}{V(R)} \int_0^R \int_0^\infty L \langle \nu(r, L) \rangle dL d^3r \equiv j(10) \left( \frac{R}{10h^{-1}Mpc} \right)^{D-3} \quad (30)$$

which is  $R$  dependent as long as the space density shows power law behavior (i.e.  $D < 3$ ). By considering  $M_* = 19.53$  (i.e.  $L_* = 1.0 \cdot 10^{10} h^{-2} L_\odot$ ),  $\alpha = -1.05$ <sup>19</sup> and by estimating the prefactor  $A$  (Eq.29) and the fractal dimension ( $D \approx 2$ ) in galaxy redshift samples we obtain<sup>48</sup>

$$j(10) \approx 2 \cdot 10^8 h L_\odot / Mpc^3. \quad (31)$$

We now estimate the density parameter in terms of the critical density,

$$\rho_c = 2.78 \cdot 10^{11} h^2 M_\odot / Mpc^3 \quad (32)$$

where  $M_\odot$  is the solar mass. By considering the product of the mass-to-luminosity ratio (in solar and  $h$  units) and the average luminosity density given by Eqs.30-31, we obtain

$$\Omega(\lambda_0) = (6 \pm 2) \cdot 10^{-4} \frac{\mathcal{M}}{h} \left( \frac{\lambda_0}{10h^{-1}} \right)^{-1}, \quad (33)$$

where  $\lambda_0$  is the scale where the crossover to homogeneity occurs (it can also be  $\lambda_0 = \infty$ , and in such a case  $\Omega(\infty) = 0$ ). Note that in view of the dependence of  $\mathcal{M}/\mathcal{L}$  on  $h$ , Eq.33 does not depend on the Hubble's constant.

Let us now suppose that  $\mathcal{M}/\mathcal{L} \approx 10h$  as it has been derived by Faber and Gallenher<sup>21</sup>. We obtain

$$\Omega(\lambda_0) \approx 6 \cdot 10^{-3} \left( \frac{\lambda_0}{10h^{-1}} \right)^{-1}. \quad (34)$$

If galaxy distribution turns out to be homogeneous at  $\lambda_0 \approx 10h^{-1}Mpc$  then  $\Omega \approx 6 \cdot 10^{-3}$  as it is obtained in the standard treatment<sup>40</sup>. If, instead, the crossover to homogeneity lies at  $100h^{-1}Mpc$ , we obtain  $\Omega \approx 6 \cdot 10^{-4}$ .

From Eq.33 and if galaxy distribution turns out to be homogeneous at  $\lambda_0$ , we obtain that the critical Mass-to-Luminosity ratio (such that  $\Omega(\lambda_0) = 1$ ) is given by

$$\left(\frac{\mathcal{M}}{\mathcal{L}}\right)_{crit} \approx 1600h \left(\frac{\lambda_0}{10h^{-1}}\right), \quad (35)$$

so that if  $\lambda_0 = 10h^{-1}Mpc$  one obtains  $(\mathcal{M}/\mathcal{L})_{crit} \approx 1600h$  (which is again consistent with the usual adopted value<sup>40</sup>) while if  $\lambda_0 = 100h^{-1}Mpc$   $(\mathcal{M}/\mathcal{L})_{crit} \approx 16000h$ , which is about two orders of magnitude larger than the highest  $\mathcal{M}/\mathcal{L}$  observed in clustered objects. For an intermediate value of  $\lambda_0 = 50h^{-1}Mpc$  one obtains  $(\mathcal{M}/\mathcal{L})_{crit} \approx 8000h$ .

For what concerns the analysis of galaxy clusters it is often used a value  $(\mathcal{M}/\mathcal{L})_c \sim 300h^{5,1}$  which, by using Eq.33, gives

$$\Omega(\lambda_0) \approx 2 \cdot 10^{-1} \left(\frac{\lambda_0}{10h^{-1}}\right)^{-1}. \quad (36)$$

Such an estimation is based on the fact that the Mass-to-Luminosity ratio found in clusters is representative of all the field galaxies. This is a very strong assumption: this means that  $(\mathcal{M}/\mathcal{L})_g = (\mathcal{M}/\mathcal{L})_c$  which is not supported by any observation. The usually adopted value of  $\Omega = 0.2^1$  can be derived from Eq.36 by assuming  $\lambda_0 = 10h^{-1}Mpc$ . In the case the crossover to homogeneity occurs at  $\lambda_0 = 100h^{-1}Mpc$  we have that  $\Omega(\lambda_0) = 0.02$  if we consider  $(\mathcal{M}/\mathcal{L})_c \sim 300h$  to be "universal".

Under the assumptions:

(i) galaxies are homogeneously distributed at scales larger than  $\lambda_0 \approx 5h^{-1}Mpc$ ,

(ii) the  $\mathcal{M}/\mathcal{L}$  of galaxies is the same of clusters, that is galaxies should contain a factor  $\sim 30$  more dark matter than what it is observed with the study of the rotation curves<sup>21</sup>,

(iii) Galaxy positions are independent of galaxy luminosity: such a assumption is not strictly valid, but it has been tested<sup>48</sup> to hold rather well in the available samples,

we get the following upper limit to  $\Omega$ . If we assume that  $\mathcal{M}/\mathcal{L} = 300h$  across all the scales, and that  $\lambda_0 \approx 50h^{-1}Mpc$ , which we consider to be a lower limit for the homogeneity scale, we get from Eq.36

$$\Omega(50Mpc/h, 300hM_\odot/L_\odot) \leq 0.04. \quad (37)$$

The direct estimation with galaxies (Eqs.33-34) gives lower a value, the reason being the strong assumption of taking  $\mathcal{M}/\mathcal{L} = 300h$  as representative of field galaxies (see Fig.3).

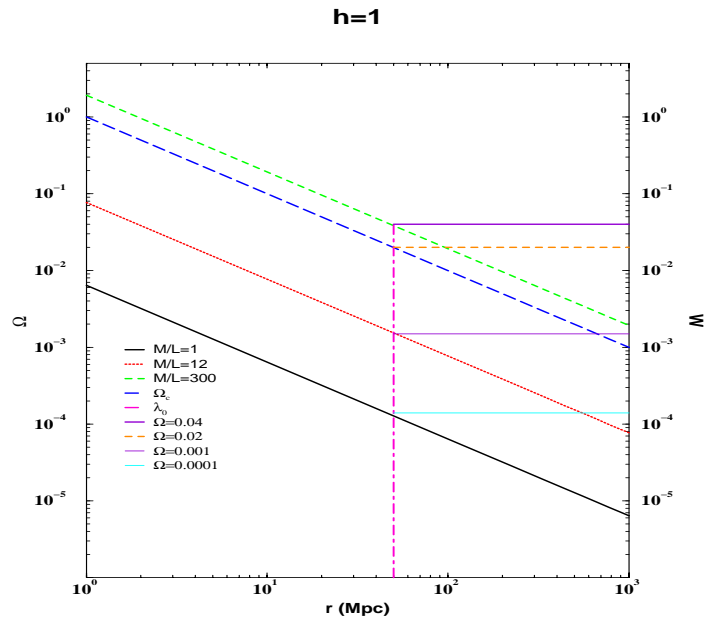


Figure 3: We show the behavior of  $\Omega(r, \mathcal{M}/\mathcal{L})$  and  $\Omega_c(r)$  (direct estimation from the Mass and number density of galaxy clusters), in the case  $h = 1$ , for three different values of  $\mathcal{M}/\mathcal{L} = 1, 12, 300$ . If  $\lambda_0 = 50h^{-1} \text{ Mpc}$  we get an upper limit  $\Omega = 0.04$ . In the figure the value of the Hubble's constant is set  $h = 1$ . (From Sylos Labini, 2000)

### 3.3 Homogeneity scale and primordial nucleosynthesis constraints

Let us now discuss the previous results in relation with the nucleosynthesis constrain  $\Omega_b^{BBN} = 0.015h^{-2}$ , deriving in such a way an upper limit for  $\lambda_0$  which is consistent with such a scenario. We may see<sup>48</sup> that if  $\lambda_0 \approx 100h^2 Mpc$  then  $\Omega(\lambda_0, (\mathcal{M}/\mathcal{L})_c) \approx \Omega_b^{BBN}$ . Such a fact has two important implications.

First of all one does not need the presence of *non baryonic dark matter* to reconcile local observations of matter contained in galaxies and galaxy clusters ( $\Omega_{local}$ ) with the primordial nucleosynthesis constraint. This is an important point as the existence of *non baryonic dark matter* has been inferred also, but not only, by the discrepancy  $\Omega_{local}/\Omega_b^{BBN}$ <sup>1</sup>, which, we show, is not observed if  $\lambda_0 \geq 100h^2 Mpc$ . There is still place, in this picture, for a uniform background of *non-baryonic dark matter*, which has completely different clustering properties from the ones of visible matter.

The second point concerns the baryon fraction in clusters. According to the usual root, in order to be consistent with  $\Omega_b^{BBN}$ , one has to force  $\Omega = 0.1 \div 0.3$  from clusters analysis<sup>1</sup>. Here we can argue as follows. By assuming a very large  $\mathcal{M}/\mathcal{L} \sim 300h$ , and that it is universal across all scales, and by assuming that all dark matter is made of baryons, if the crossover to homogeneity occurs at scales larger than  $\sim 150h^2 Mpc$  then there is not enough baryonic matter to satisfy the nucleosynthesis constraint. The situation gets clearly worst if one takes into account that not all the mass in clusters is baryonic or that  $\mathcal{M}/\mathcal{L} < 300h$ , i.e. it is not considered to be universal across all scales, or that  $\lambda_0 \gg 100h^2 Mpc$ . There are then two different possibilities: the first is to study the case of a non homogeneous primordial nucleosynthesis which could lower the limit on  $\Omega_b^{BBN}$ , given the observed abundances of light elements, and the second would be to have a more uniform background of baryonic dark matter. This seems rather unlikely because it would have very different clustering properties with respect to visible mass and it is very difficult to find a dynamical explanations for such a segregation.

### 3.4 Where does linear approximation hold ?

In the standard picture, the properties of dark matter on cosmological relevant scales  $r > 5h^{-1} Mpc$ , are inferred from the observed properties of visible matter and radiation. Now one studies change in these properties, i.e. the presence of fractal correlations, and in this respect they will have consequences on dark matter too<sup>18,26</sup>. For example, the determination of the mass density including dark matter has been performed on the basis of the linear theory<sup>43</sup>. Here the problem is: beyond which scale can linear theory be considered as a useful approximation ?

In other words, the dynamical estimates use gravitational effects, of departure from a strictly homogeneous distribution on the motion of objects such as galaxies considered as a test particle. A completely different situation occurs if  $\lambda_0$  is larger than the scales at which linear approximation is usually adopted. For example, methods based on the Cosmic Virial Theorem<sup>39</sup>, distortions in redshift surveys<sup>43</sup>, local group dynamics<sup>40</sup>, the "reconstructed" peculiar velocity field from the density field (e.g POTENT-like methods<sup>43</sup>), are clearly not useful at scales  $r \ll \lambda_0$ . Up to now it is implicitly assumed  $\lambda_0 \approx 5 \div 10 h^{-1} Mpc$  and all these methods are considered to be valid on larger scales. If, for example  $\lambda_0 \sim 50 h^{-1} Mpc$  then it is not possible to interpret peculiar velocities in the range  $1 \div 30 h^{-1} Mpc$  through the linear approximation (as it is usually done<sup>43</sup>), unless there is a background of dark matter which is uniform beyond a certain small scale. In such a situation however the estimates would be different<sup>2</sup> from the usual ones<sup>43</sup>.

Let us review some simple relations between the conditional average density  $\bar{\rho}(r)$  and the usual correlation function  $\xi(r)$ . If  $\bar{\rho}(r)$  has a power law behavior up to a scale  $\lambda_0$  and then it presents a crossover to homogeneity, it is simple to show that (in the case of  $D = 2$ ) the length scale at which  $\xi(r_0) = 1$  is of the order of

$$r_0 \approx \frac{\lambda_0}{3} \quad (38)$$

where the exact relation between  $r_0$  and  $\lambda_0$  depends on the details of the crossover<sup>26</sup>. However Eq.38 gives a reasonable order of magnitude in the general case. In such a situation it is also simple to compute  $\sigma(r, \lambda_0) = \langle N(r) - \langle N \rangle \rangle / \langle N \rangle$ , i.e the amplitude of the fluctuation with respect to the average at the scale  $r \ll \lambda_0$ <sup>46</sup>

$$\sigma(r, \lambda_0) \approx \frac{\lambda_0}{2r}. \quad (39)$$

It has been found in various nearby surveys that  $\sigma(8h^{-1}Mpc) = 1$ . However if the crossover to homogeneity occurs at  $\lambda_0$  we have that

$$\sigma(8h^{-1}Mpc, \lambda_0) \approx \frac{\lambda_0}{16} \quad (40)$$

For example if  $\lambda_0 \approx 15h^{-1}Mpc$  then  $r_0 \approx 5h^{-1}Mpc$  and  $\sigma(8h^{-1}Mpc, 15h^{-1}Mpc) = 1$ ; otherwise if  $\lambda_0 \approx 50h^{-1}Mpc$  then  $r_0 \approx 16h^{-1}Mpc$  and  $\sigma(8h^{-1}Mpc, 15h^{-1}Mpc) = 3$ .

Linear approximation holds in the linear regime, when the amplitude of the density contrast is small, i.e. for  $\sigma \ll 1$ . We have that  $\sigma = 1$  at a scale  $r_{\sigma=1}(\lambda_0)$

$$r_{\sigma=1}(\lambda_0) \approx \frac{\lambda_0}{2} \quad (41)$$

For instance, if the crossover to homogeneity occurs at  $\lambda_0 = 50h^{-1}Mpc$  one has that  $\sigma(\lambda_0) = 1$  at  $r_{\sigma=1} = 25h^{-1}Mpc$ . In such a situation all the estimations of the density parameter at smaller scales based on the linear approximation<sup>43</sup>, and hence based on the untested assumption of linearity, are not correct.

We have studied in detail<sup>24</sup> the gravitational force distribution in a fractal structure. Its behaviour can be understood as the sum of two parts, a local or ‘nearest neighbours’ piece due to the smallest cluster (characterised by the lower cut-off  $\Lambda$  in the fractal) and a component coming from the mass in other clusters. The latter is bounded above by the scalar sum of the forces

$$\langle |\vec{F}| \rangle \leq \lim_{L \rightarrow \infty} \int_{\Lambda}^L \frac{G\rho_m(r)}{r^2} 4\pi r^2 dr \sim L^{D-2} \quad (42)$$

so that for  $D < 2$  it is convergent, while for  $D > 2$  it may diverge. If there is a divergence, it is due to the presence of angular fluctuations at large scales, described by the three-point correlation properties of the fractal. For the difference in the force between two points the local contribution will be irrelevant well beyond the scale  $\Lambda$ , while it is easy to see that the ‘far-away’ contribution will now converge as  $L^{D-3}$ , and its being non-zero is a result of the absence of perfect spherical symmetry. We have then applied such a result to the case of an open universe<sup>31</sup> in order to compute the expected deviations from a pure linear Hubble flow.

#### 4 Discussion and Conclusions

We now present a short discussion about the perspective of our work: the cosmological implications of the fractal behavior of visible matter crucially depend on the crossover scale  $\lambda_0$ , but, no matter what is the actual value of such a scale, we have some important consequences from the theoretical point of view. We may identify three different scenarios.

- (1) The fractal extends only up to  $\sim 30 \div 100h^{-1}Mpc$ . This is the minimal concept which begins to be absorbed in the literature<sup>49</sup> but, sometimes, without considering its real consequences. The standard approach to galaxy distribution has identified very small “correlation lengths”, namely  $5h^{-1}Mpc$  for galaxies and  $25h^{-1}Mpc$  for clusters. These numbers (which were supposed to know with high precision<sup>14</sup>) are anyhow inconsistent with the fractal extending a factor  $2 \div 4$  more. We have shown that this inconsistency is conceptual and not due to incomplete data or weak statistics. Hence, in this hypothesis one has to abandon all the concepts related to these length scales. These are:

- (i) The estimate of the matter density in clusterised objects (visible + dark), which has been claimed to be  $\Omega \approx 0.2 \div 0.3$ , decreases by one order of magnitude or more.
- (ii) The normalization of N-body simulations is usually performed to some length-scale or amplitude of fluctuations, which are related to  $5h^{-1}Mpc$  and  $25h^{-1}Mpc$ .
- (iii) Concepts like the galaxy-cluster mismatch and the related luminosity bias, as well as the understanding of the clustering via the bias parameter  $b$  (i.e. linear or non-linear - "stochastic bias" - amplification of  $\xi(r)$ ) loose any physical meaning.
- (iv) The interpretation of the velocity field is also based on the linear approximation which cannot certainly hold at scales smaller than  $30 \div 50h^{-1}Mpc$ .
- (v) The reconstruction of the three dimensional properties from the angular data suffers of the same untested assumption of homogeneity.

In summary major modifications are necessary for the origin and dynamics of large scale structures and for the role of dark matter. However the structures may still formed via gravitational instability, in the sense that they are not necessarily primeval<sup>31</sup>.

- (2) The fractal extends up to  $300 \div 500h^{-1}Mpc$ . In this case the standard picture of gravitationally induced structures after the electromagnetic decoupling is untenable. There is no time to create such large scale correlated structures via gravitational instability, starting from Gaussian initial conditions. More string consequences are clearly important for what concerns the amount of matter in clusterised objects.
- (3) The fractal extends up to  $\gtrsim 1000h^{-1}Mpc$ , and homogeneity does not exist, at least for what concerns galaxies. In this extreme case a new picture for the global metric<sup>31</sup> is then necessary.

For some questions the fractal structure leads to a radically new perspective and this is hard to accept. But it is based on the best data and analyses available. It is neither a conjecture nor a model, it is a fact. The theoretical problem is that there is no dynamical theory to explain how such a fractal Universe could have arisen from the pretty smooth initial state we know existed in the big bang. However this is a different question. The fact that something can be hard to explain theoretically has nothing to do with whether it is true or not. Facing a hard problem is far more interesting than hiding it under the rug by an inconsistent procedure. For example some interesting attempts



to understand why gravitational clustering generates scale-invariant structures have been recently proposed by de Vega et al<sup>15,16,17</sup>. Indeed this will be the key point to understand in the future, but first we should agree on how these new 3d data should be analyzed. In addition, the eventual crossover to homogeneity has also to be found with our approach. If for example homogeneity would really be found say at  $\sim 100h^{-1}Mpc$ , then clearly all our criticism to the previous methods and results still holds fully. In summary the standard method cannot be used neither to disprove homogeneity, nor to prove it. One has simply to change methods.

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